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Wall layers with non-uniform shear stress

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The empirical description of turbulent wall layers across which the shear stress varies is considered. The description given by Townsend for zero-stress layers is found to be inapplicable to uniform pressure flows in pipes and two-dimensional channels, and to a boundary layer developing in the absence of a pressure gradient.

1. Introduction

Townsend (1961*a*) has generalized Prandtl's description of turbulent flow near a smooth wall

$$\frac{\partial U}{\partial y} = \frac{(\tau/\rho)^{\frac{1}{2}}}{Ky}, \tag{1}$$

obtaining

$$\frac{\partial U}{\partial y} = \frac{(\tau/\rho)^{\frac{1}{2}}}{Ky}\left[1 - B\frac{y}{\tau}\left|\frac{\partial \tau}{\partial y}\right|\right], \tag{2}$$

where $\partial U/\partial y$ is the gradient of the time-mean velocity, τ is the time-mean local shear stress, y is the distance measured from the wall, $K \simeq 0{\cdot}40$ is the fundamental von Kármán constant, and B is an absolute constant characterizing the lateral diffusion of turbulence energy through the wall layer. Consideration of the data describing a self-preserving zero-stress layer led to the prediction that the constant $B \simeq 0{\cdot}18$ (Townsend 1961*b*).

We shall attempt to obtain Townsend's result (2) from dimensional considerations alone, without discussing the structure of the turbulence within the wall layer. Then empirical data relating to simple channel flows will be introduced to determine the constant B for these cases.

2. Dimensional arguments

We restrict consideration to turbulent wall layers that are uniform in the direction of a unidirectional mean flow, such as would arise ultimately in long circular tubes or two-dimensional channels. In these cases there must exist in the fully turbulent part of the flow a relationship of the form (Townsend 1956)

$$dU/dy = f(\tau_0, \rho, h, y)$$

among the variables describing the motion. Here τ_0 is the shear stress at the wall, and h is a length characterizing the width of the flow, half the channel width for two-dimensional flows, and the pipe radius for pipe flows.

These parameters are related by

$$\tau = \tau_0(1-y/h) \quad \text{or} \quad d\tau/dy = \alpha = -\tau_0/h,$$

so that the relationship may be rewritten

$$dU/dy = f_1(\tau, \rho, \alpha, y),$$

a more convenient starting point for the present discussion. Dimensional homogeneity requires that

$$\frac{y}{(\tau/\rho)^{\frac{1}{2}}}\frac{dU}{dy} = f_2\left(\frac{y\alpha}{\tau}\right).$$

Expressing this result as a simple power series (as seems permissible since Prandtl's basic result (1) gives a good description of the flow near the wall), we may write

$$\frac{y}{(\tau/\rho)^{\frac{1}{2}}}\frac{dU}{dy} = a_0 + a_1\left(\frac{y\alpha}{\tau}\right) + a_2\left(\frac{y\alpha}{\tau}\right)^2 + \dots. \tag{3}$$

This result, truncated to two terms, is formally just that (2) given by Townsend (1961*a*). His analysis seems equivalent to the following.

The starting point is the energy equation valid in the fully turbulent part of the motion:

$$\overline{uv}\frac{dU}{dy} + \frac{d}{dy}\left[\overline{v\left(\frac{p}{\rho} + \tfrac{1}{2}u_i^2\right)}\right] + \epsilon = 0,$$

where u, v, u_i, p are fluctuations in the turbulent flow, and ϵ gives the dissipation of energy. Note that $\overline{uv} = -\tau/\rho$ very nearly, in this part of the flow. We find also that

$$\overline{v(p/\rho + \tfrac{1}{2}u_i^2)} = b(\tau/\rho)^{\frac{3}{2}} \quad \text{and} \quad \epsilon = (a/y)(\tau/\rho)^{\frac{3}{2}}$$

with a and b dimensionless constants, on the assumption that these quantities depend on τ, y, and ρ but not on α. The energy equation then gives

$$\frac{dU}{dy} = \frac{a}{y}\left(\frac{\tau}{\rho}\right)^{\frac{1}{2}}\left[1 - \frac{3b}{2a}\frac{y}{\tau}\frac{d\tau}{dy}\right],$$

which may be rewritten in the forms (2) or (3).

3. Empirical information

We shall now determine the constant a_1 which gives the primary effect of varying shear, not by examining the extreme case of a zero-stress layer, but by a study of simple channel flows and boundary layers. To do this in a simple way, we introduce the mixing length l defined by

$$\frac{dU}{dy} = \frac{1}{l}\left(\frac{\tau}{\rho}\right)^{\frac{1}{2}}.$$

(Note that the length defined here is used simply as a means of presenting data compactly; no special physical importance is to be imputed to it.) Now the series (3) may be rewritten

$$y/l = a_0 - a_1(y/h) + \dots \quad \text{since} \quad y/h = -y\alpha/\tau_0.$$

(1) *Circular tubes*

Schlichting (1955, p. 408) quotes the result of Nikuradse's empirical determination of the variation of mixing length across a circular tube,

$$l/y = 0{\cdot}40 - 0{\cdot}44\,y/h + 0{\cdot}24(y/h)^2 - 0{\cdot}06(y/h)^3.$$

Then $a_0 = 2{\cdot}5$, and $a_1 = -2{\cdot}75$ for this flow.

(2) *Two-dimensional channels*

So far as is known, the data for this case are not available in the convenient form given above for circular tubes. The mean velocity and shear stress measurements of Laufer (1950) have been analysed to obtain this information. The calculated values of the mixing length are shown in figures 1 and 2.

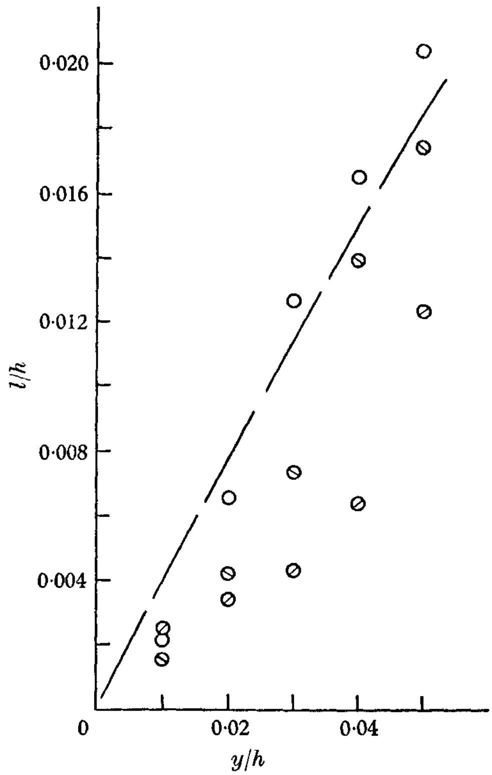

FIGURE 1. Variation of mixing length with distance from the wall in two-dimensional channel flow. Data from Laufer (1950). Channel Reynolds number: ⊘ $R = 12{,}300$, ⦸ $R = 30{,}800$, ○ $R = 61{,}600$.

Only the part of the data describing fully-turbulent flow is relevant to our study. If we take

$$\delta = 50\nu/(\tau_0/\rho)^{\frac{1}{2}} \quad \text{or} \quad \delta/h = 50/R(\tau/\rho U_0^2)^{\frac{1}{2}} \quad \text{with} \quad R = U_0 h/\nu,$$

to give the inner boundary of the turbulent region (see Schlichting 1955, p. 407) we can use Laufer's measurements (table 1) to determine the limits of applicability of the data. Note that U_0 is the velocity at the centre line of the channel.

It appears that only the highest few points shown in figure 1 are relevant to the empirical relationship sought.

Turning to figure 2 we see that the scatter is very great for high values of y/h; it was impossible to determine the small velocity gradients accurately from the data available. However, in the intermediate range the data is consistent enough for our purpose.

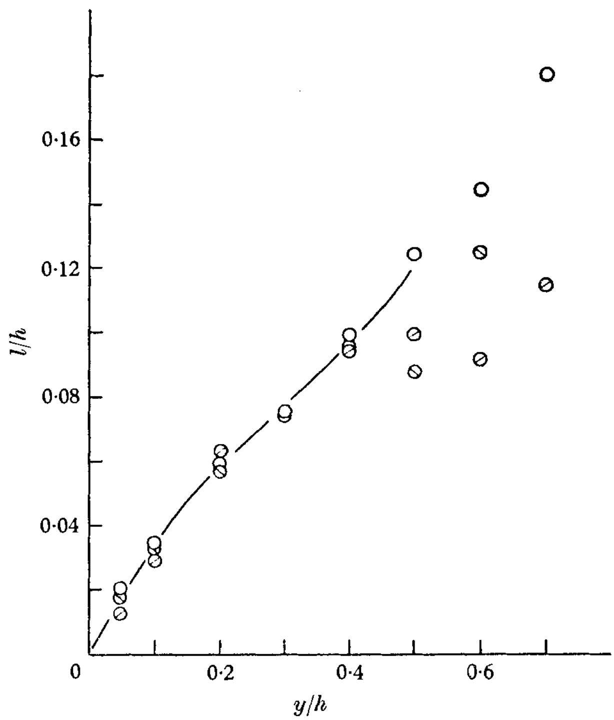

FIGURE 2. See caption of figure 1.

R	$(\tau/\rho U_0^2)^{\frac{1}{2}}$	δ/h
12,300	0·042	0·096
30,800	0·038	0·043
61,600	0·037	0·022

TABLE 1.

We take $a_0 = 2{\cdot}5$ as the basis of the calculation, a choice which will be justified *a posteriori*, and require in addition that the empirical curve pass through the points $(y/h, l/h)$ equal to $(0{\cdot}20, 0{\cdot}058)$ and $(0{\cdot}40, 0{\cdot}096)$. Then

$$l/y = 0{\cdot}4 - 0{\cdot}7y/h + 0{\cdot}75(y/h)^2.$$

This curve is consistent with the relevant data of figure 1 as well. For $y/h > 0{\cdot}5$, it gives quite unrealistic values. We find that $a_1 = -4{\cdot}37$ for two-dimensional channel flow.

(3) *Boundary layer in absence of pressure gradient*

From the data describing the variation of time-mean velocity through a turbulent boundary layer (Townsend 1956), we choose the points $(y/\delta_0, l_0/K\delta_0)$ equal to (0·2, 0·166) and (0·4, 0·242) to represent the variation of mixing length near the wall. Then

$$l_0/Ky = 1 - 0{\cdot}713\,y/\delta_0 - 0{\cdot}69(y/\delta_0)^2$$

is an approximate expression for the mixing length. Here

$$l_0 = (\tau_0/\rho)^{\frac{1}{2}}/(dU/dy)$$

is a mixing length based on the shear stress near the wall.

The stress variation across the boundary layer is given quite accurately by

$$\tau = \tau_0[1 - 0{\cdot}733\,y/\delta_0] \quad \text{for} \quad y/\delta_0 < 1{\cdot}0.$$

Then

$$l = l_0(\tau/\tau_0)^{\frac{1}{2}} = l_0[1 - 0{\cdot}733\,y/\delta_0]^{\frac{1}{2}}$$

and to the first order

$$l/Ky = 1 - 1{\cdot}08y/\delta_0.$$

But for the linear shear stress variation $\delta_0 = -0{\cdot}733\tau_0/\alpha$, so that

$$Ky/l = 1 - 1{\cdot}47\alpha y/\tau_0$$

to the first order in the parameter of expansion. Taking $a_0 = 1/K = 2{\cdot}5$, as before, we find $a_1 = -3{\cdot}68$, while $B = 1{\cdot}47$.

4. Conclusions

The values of the parameters a_1 and B applicable to the several flows considered are given in table 2. Two conclusions emerge from this tabulation.

Flow	a_1	B
Zero-stress boundary layer	−0·45	0·18
Circular pipe	−2·75	1·10
Two-dimensional channel	−4·37	1·75
Boundary layer with zero pressure gradient	−3·68	1·47

TABLE 2.

(1) There is no universal equilibrium layer, save for the familiar logarithmic layer in which the mixing length varies linearly with distance from the wall. We conclude that the influence of the core flow (which undoubtedly differs in structure from flow to flow) extends so far into the wall layer that no region exists in which the stress gradient is a dominant parameter.

(2) For the most common wall flows, the parameter $B = 1{\cdot}5 \pm 0{\cdot}4$, a value differing by nearly an order-of-magnitude from that characterizing the zero-stress layer. However, this parameter varies little enough among flows with large wall stress, that the assumption of a constant value ($B \simeq 1{\cdot}5$) for a wide variety of wall layers could be valid for some calculations.

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